EXERCISESET 13, TOPOLOGY IN PHYSICS

- The hand-in exercise is exercise 3.
- Please hand it in electronically at `topologyinphysics2019@gmail.com` (1 pdf!)
- Deadline is Wednesday May 15, 23.59.
- Please make sure your name and the week number are present in the file name.

EXERCISES

Exercise 1: The Signature. Suppose M is a compact and oriented manifold of dimension n. This means we have bilinear pairings

$$H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR}}(M) \longrightarrow \mathbb{C}$$

for all $0 \leq k \leq n$ given by

$$\langle [\alpha], [\beta] \rangle = \int_M \alpha \wedge \beta.$$

If we suppose further that $n = 4q$ for some $q \geq 0$ we obtain in particular a bilinear pairing on $H^{2q}_{\mathrm{dR}}(M)$, we will denote this pairing P. Given a bilinear pairing η on any vector space V we can associate the number $sign(\eta)$ given by

$$\#\{1 \leq j \leq \operatorname{Dim} V \mid \eta(\alpha_j, \alpha_j) > 0\} - \#\{1 \leq j \leq \operatorname{Dim} V \mid \eta(\alpha_j, \alpha_j) < 0\}$$

where $\alpha_1, \dots, \alpha_{\operatorname{Dim} V}$ is a basis of V. It is an elementary fact from linear algebra that this number does not depend on the basis, but only on η. The *signature* of M denoted $sign(M)$ is given by the signature $sign(P)$ of P and constitutes an invariant of the manifold M. In this exercise you will show that the signature of M is computed as $\operatorname{Tr}_s e^{-\beta H}$ for $H = Q^2$, where $Q = d + d^*$ (which is clearly elliptic).

Consider the grading $\Omega^\bullet(M) = H^0 \oplus H^1$ where $H^0 = \Omega^{even}(M)$ and $H^1 = \Omega^{odd}(M)$ and equip M with some Riemannian structure g. Recall the Hodge star operation $\star$ and the fact that $\star^2 \alpha = (-1)^{np+p}\alpha$ on p-forms α. Also set

$$\delta\alpha = (-1)^{np+n+1} \star d \star \alpha.$$

a) Show that $Q = d + \delta$ is an odd and self-adjoint operator.
b) Show that $\operatorname{Ker} Q = \operatorname{Ker} d \cap \operatorname{Ker} d^*$ (note that $d^* = \delta$).

In the following you may use the Hodge theorem:

$$\Omega^p(M) = \operatorname{Ker} d \oplus \operatorname{Im} d^* = \operatorname{Im} d \oplus \operatorname{Ker} d^*.$$

c) Show that $\operatorname{Ker} Q \simeq \bigoplus_{p \geq 0} H^p_{\mathrm{dR}}(M)$.

d) Recall the defnition of the Euler characteristic

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \mathrm{Dim}\, H^i_{\mathrm{dR}}(M).$$

Show that

$$\mathrm{Tr}_s e^{-\beta Q^2} = \chi(M).$$

Consider the operator ϵ on $\Omega^\bullet(M)$ given by

$$\epsilon\alpha = i^{2q+p(p-1)} \star \alpha$$

for α a p-form.

e) Show that ϵ is self-adjoint and $\epsilon^2 = 1$.

Now Consider the grading $\Omega^\bullet(M) = H^+ \oplus H^-$ where H^+ is the $+1$ eigenspace of ϵ and H^- is the -1 eigenspace of ϵ.

f) Show that Q is still odd for the new grading.

g) Show that $\mathrm{Tr}_s e^{-\beta Q^2} = sign(M)$ if we consider Tr_s with respect to the new grading.

HINT: Consider the decomposition into ϵ invariant spaces

$$\mathrm{Ker}\, Q = (H^0 \oplus H^n) \oplus (H^1 \oplus H^{n-1}) \oplus \ldots \oplus (H^{2q-1} \oplus H^{2q+1}) \oplus H^{2q}$$

given by $H^p = \{\alpha \in \Omega^p(M) \mid d\alpha = d^*\alpha = 0\}$. Each of these components contributes seperately to the supertrace. Now use the fact that $P([\alpha],[\beta]) = \langle \alpha, \star\beta \rangle$ to show that the final term gives the signature. Use the fact that for each $+1$ eigenvector $\alpha + \epsilon(\alpha) \in H^p \oplus H^{n-p}$ we can find the -1 eigenvector $\alpha - \epsilon(\alpha)$ to show that the other components do not contribute to the supertrace.

Note that in the above we have computed two different quatities from the same operator. Thus the grading applied really is vitally important. This shows also from the index theorem, since the characteristic classes appearing in it are determined also by grading. In the first case we would obtain the Euler class. The term to be integrated to obtain the signature is the so-called (Hirzebruch) L-class.

Exercise 2: The groups $\mathrm{Spin}(n)$**.** Recall that inside the Clifford algebra Cliff_n we have

$$\mathrm{Pin}(n) := \{\psi(x_1)\cdots\psi(x_k),\ ||x_i||^2 = 1,\ \text{for all } i = 1,\ldots,k\},$$
$$\mathrm{Spin}(n) := \{\psi(x_1)\cdots\psi(x_k) \in \mathrm{Pin}(n),\ k \text{ is even}\}.$$

a) Show that, equipped with Clifford multiplication, $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$ are groups.

b) Show that the map $\epsilon : \mathrm{Cliff}_n \to \mathrm{Cliff}_n$ defined by

$$\epsilon(\psi_{i_1}\cdots\psi_{i_k}) = \begin{cases} \psi_{i_1}\cdots\psi_{i_k} & k \text{ even}, \\ -\psi_{i_1}\cdots\psi_{i_k} & k \text{ odd}. \end{cases}$$

is well-defined, and turns Cliff_n into a *superalgebra*: define $\mathrm{Cliff}_n^{\pm}$ as the ± 1 eigenspace of ϵ, and show that the multiplication satisfies

$$\mathrm{Cliff}_n^{\pm}\mathrm{Cliff}_n^{\pm} \subset \mathrm{Cliff}_n^{+} \quad \text{and} \quad \mathrm{Cliff}_n^{\mp}\mathrm{Cliff}_n^{\pm} \subset \mathrm{Cliff}_n^{-}.$$

c) Define a map $\rho : \mathrm{Pin}(n) \to O(n)$ as follows: for all $g \in \mathrm{Pin}(n)$, $\rho(g)$ is the unique element of $O(n)$ such that

$$g\psi(v)\epsilon(g^{-1}) = \psi(\rho(g)v), \quad \text{for all } v \in \mathbb{R}^n.$$

Show that for $g = \psi(x)$, $||x||^2 = 1$, $\rho(g)$ is a reflection in a hyperplane.

d) Using the well-known fact that any orthogonal transformation can be written as a product of reflections, show that ρ defines a surjective group homomorphism. Show that $\ker(\rho) = \mathbb{Z}_2$. This shows that we have defined a short exact sequence of groups:

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}(n) \to O(n) \to 1.$$

e) Show that there is a similar exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \to SO(n) \to 1.$$

f) Suppose that $n \geq 2$, and choose two orthonormal vectors $e_1, e_2 \in \mathbb{R}^n$. Consider the path

$$\gamma(s) = (\psi(e_1)\sin(s) + \psi(e_2)\cos(s))(\psi(e_1)\sin(s) - \psi(e_2)\cos(s)), \quad s \in [0, \frac{\pi}{2}],$$

and show that it gives a path from $+1$ to -1. In what way does this show that the covering in $e)$ is *nontrivial*?

$\star$ **Exercise 3: Spinors.** Consider the Clifford algebra Cliff_{2n} of the Euclidean vector space $\mathbb{R}^{2n}$. In this exercise we will construct the *spinor representation* using creation and annihilation operators.

a) Define the fermionic creation and annihilation operators by

$$a_k := \frac{1}{2}(\psi_k + i\psi_{k+n}), \quad a_k^{\dagger} := \frac{1}{2}(\psi_k - i\psi_{k+n}), \quad k = 1, \ldots, n.$$

Compute the anticommutators

$$[a_k, a_l], \quad [a_k^{\dagger}, a_l^{\dagger}], \quad [a_k, a_l^{\dagger}].$$

b) Construct a Hilbert space $\mathcal{H}$ by letting the $a_k^{\dagger}$'s act on a ground state $|0\rangle$ by creation operators and defining $a_k^{\dagger}$ to be the adjoint of a_k. What is the dimension of $\mathcal{H}$?

c) We consider now the case $n = 1$. Show that any element in $\mathrm{Spin}(2)$ can be written as

$$g(\theta) := \cos\theta + \sin\theta\,\psi_1\psi_2,$$

so that topologically, $\mathrm{Spin}(2)$ is a circle. What is the image of $g(\theta)$ in $SO(2) \cong U(1)$ under the map ρ?

d) Compute the action of $g(\theta)$ on $|0\rangle \in \mathcal{H}$ and on $|1\rangle := a_1^\dagger |0\rangle$.

Exercise 4: Index on an odd dimensional manifold. In this exercise we will compute the index of differential operators on odd dimensional manifolds. So let us fix the manifold M of dimension $m = 2n+1$, vector bundles E and F over M and a degree k elliptic differential operator $D\colon \Gamma(M; E) \to \Gamma(M; F)$.

a) Recall that the (principal) symbol σ_D can be viewed as a section of the bundle $\pi^*\mathrm{Hom}(\mathrm{E}, \mathrm{F})$ over T^*M. Show that it is homogeneous of order k in the sense that if $\phi_\lambda\colon T^*M \to T^*M$ denotes the fiberwise scaling by a factor $\lambda \in \mathbb{R}$, then

$$\phi_\lambda^* \sigma_D = \phi_{\lambda^k} \circ \sigma_D$$

where in the second instance ϕ_{λ^k} denotes fiberwise scaling by a factor λ^k in the bundle $\pi^*\mathrm{Hom}(\mathrm{E}, \mathrm{F})$.

HINT: Note that locally the statement is simply $\sigma_D(x, \lambda\xi) = \lambda^k \sigma_D(x, \xi)$.

A rank r vector bundle $P\colon R \to M$ is called oriented if each fiber R_x carries an orientation in a compatible way. This means that there exists a differential form $\tau \in \Omega^r(R)$ such that $\int_{R_x} \tau = 1$ for all $x \in M$. The *Thom isomorphism theorem* states that in this case we have

$$H_c^\bullet(R) \simeq H^{\bullet - r}(X)\tau.$$

Here the subscript c refers to the fact that we only consider compactly supported differential forms in the definition of cohomology. This means that each class α in $H_c(R)$ can be written uniquely as $\alpha = P^*\beta \wedge \tau$. The first version of the index formula was in fact expressed as

$$\text{Index } D = \int_{T^*M} \mathrm{ch}(\sigma_D)\mathrm{Td}(M)$$

and subsequently rewritten using the Thom isomorphism. Here $\mathrm{ch}(\sigma_D)$ denotes the class given by the elliptic complex

$$0 \to \pi^*E \xrightarrow{\sigma_D} \pi^*F \to 0.$$

b) Show that $\mathrm{ch}(\sigma_D) = \mathrm{ch}(\alpha^*\sigma_D)$ where $\alpha = \phi_{-1}$.

c) Show that $\alpha^*\tau = (-1)^m \tau$

d) Show that Index $D = 0$.